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Letter to the Editor

Residual motion in damped linear systems

S.M. Shahruz[a,*], P. Kessler[b]

[a] Berkeley Engineering Research Institute, P.O. Box 9984, Berkeley, CA 94709, USA
[b] Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

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1. Introduction

The equations of motion of damped linear n-degree-of-freedom dynamical systems can be represented by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = \theta_n, \quad x(0) =: x_0, \ \ \dot{x}(0) =: \dot{x}_0 \tag{1}$$

for all $t \geqslant 0$, where

$$x(t) = [x_1(t) \ x_2(t) \ \dots \ x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n \tag{2}$$

is the displacement vector, $x_0 \in \mathbb{R}^n$ and $\dot{x}_0 \in \mathbb{R}^n$ are, respectively, the vectors of initial displacements and velocities, θ_n is the zero vector in $\mathbb{R}^n$; the mass matrix $M \in \mathbb{R}^{n\times n}$ and the stiffness matrix $K \in \mathbb{R}^{n\times n}$ are symmetric and positive definite, and the damping matrix $C \in \mathbb{R}^{n\times n}$ is symmetric and non-negative definite.

Let the damping matrix C be positive semi-definite. In this case, some components of system (1) lack damping. Thus, it can happen that $x_j(t)$ for some $j = 1, 2, \dots, n$ would oscillate persistently without decaying to zero as $t \to \infty$. In this case, system (1) is said to have *residual motion*. Also, when C is positive semi-definite, it can happen that $x_j(t) \to 0$ for all $j = 1, 2, \dots, n$ as $t \to \infty$. The latter situation is somewhat surprising since the positive semi-definiteness of C implies that some components of system (1) lack damping, and hence they would conceivably oscillate persistently without coming to rest. Thus, a problem to be solved is as follows.

Problem P. In system (1), let the damping matrix C be positive semi-definite. Under what conditions does system (1) have or not have residual motion?

The authors of Refs. [1–7] have attempted to solve Problem P. The authors of Refs. [1,2,4,6] have used intuitive arguments to establish the existence or non-existence of residual motion for

*Corresponding author. Tel.: +1-510-642-3248.
E-mail address: shahruz@newton.berkeley.edu (S.M. Shahruz).

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two-, three-, and four-degree-of-freedom systems. In Refs. [3,5,7], the existence of residual motion is established by matrix rank tests.

In this note, Problem P is solved by giving easy-to-check conditions which establish the existence or non-existence of residual motion in system (1).

2. Residual motion

In this section, the residual motion in system (1) is studied.

System (1) is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \tag{3}$$

for all $t \geqslant 0$, where

$$A := \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{4}$$

and I_n denotes the $n \times n$ identity matrix. The solution of system (3) is given by

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \exp(At) \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \tag{5}$$

for all $t \geqslant 0$.

The unique equilibrium point of system (3) is $X_e := [\theta_n^{\mathrm{T}} \; \theta_n^{\mathrm{T}}]^{\mathrm{T}}$. The equilibrium point X_e is said to be *asymptotically stable* if and only if $x_j(t) \to 0$ and $\dot{x}_j(t) \to 0$ for all $j = 1, 2, \ldots, n$ as $t \to \infty$. Thus, the asymptotic stability of X_e is equivalent to the non-existence of residual motion in system (1). It is well known that X_e is asymptotically stable if and only if A is a Hurwitz matrix, i.e., all eigenvalues of A are in the open left-half of the complex plane, denoted by $\mathbb{C}_-^o$; see. e.g., Ref. [8, p. 103] or [9, Theorem 33, p. 195].

A result is now stated which is relevant to residual motion.

Theorem 2.1. *In system* (1), *if C is a positive definite matrix*, *then the system does not have residual motion*; *equivalently*, *A in Eq.* (4) *is a Hurwitz matrix.*

Proof. By a result in Ref. [8, p. 123], if C is a positive definite matrix, then X_e is the asymptotically stable equilibrium point of system (3). Thus, there is no residual motion. The asymptotic stability of X_e is equivalent to having A a Hurwitz matrix. □

The interesting case of positive semi-definite damping matrix C is now considered. A useful fact is first stated.

Fact 2.2. *If C is a positive semi-definite matrix*, *then eigenvalues of the matrix A in Eq.* (4) *are in $\mathbb{C}_-^o$ or on the imaginary axis of the complex plane.*

Proof. See Ref. [10, Theorem 3, p. 246]. □

Now, the existence or non-existence of residual motion in system (1) is established.

Theorem 2.3. *In system* (1), *let C be a positive semi-definite matrix. System* (1) *does not have residual motion if and only if A in Eq.* (4) *is a Hurwitz matrix.*

Proof. Obvious: the equilibrium point X_e of system (3) is asymptotically stable if and only if A is a Hurwitz matrix; see, e.g., Ref. [8, p. 103] or [9, Theorem 33, p. 195]. □

Remark 2.4. Theorem 2.3 provides a test by which the existence or non-existence of residual motion in system (1) can be established. This test requires the computation of all $2n$ eigenvalues of the matrix $A \in \mathbb{R}^{2n \times 2n}$. Once the eigenvalues of A are computed the existence or non-existence of residual motion in system (1) is decided. By Fact 2.2, it is guaranteed that A does not have eigenvalues with positive real parts. If all eigenvalues of A have negative real parts, then system (1) does not have residual motion. If A has pairs of eigenvalues on the imaginary axis of the complex plane, then system (1) has residual motion. Note that multiple eigenvalues of A of any multiplicity on the imaginary axis of the complex plane do not cause instability in the system (unbounded solution). This statement is proved as follows. Let

$$E(t) = \tfrac{1}{2}[x^{\mathrm{T}}(t)\ \dot{x}^{\mathrm{T}}(t)] \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \tag{6}$$

for all $t \geqslant 0$. It is straightforward to show that along the solution of system (3), $\dot{E}(t) = -\dot{x}^{\mathrm{T}}(t) C \dot{x}(t)$ for all $t \geqslant 0$. Since C is positive semi-definite, it follows that $\dot{E}(t) \leqslant 0$ for all $t \geqslant 0$. Thus, $t \mapsto E(t)$ is a bounded (non-increasing) function of time; so are $t \mapsto x(t)$ and $t \mapsto \dot{x}(t)$.

Next, conditions are sought by which the existence or non-existence of residual motion in system (1) can be established without computing the eigenvalues of the matrix A. Before stating such conditions, some preliminary results are given.

A non-singular matrix, known as modal matrix, is used in the modal analysis of system (1). Let $U \in \mathbb{R}^{n \times n}$ denote the modal matrix. The columns of U are the eigenvectors of the symmetric generalized eigenvalue problem

$$K u_j = \omega_j^2 M u_j \tag{7}$$

for all $j = 1, 2, \ldots, n$, where $\omega_j^2 > 0$ and $u_j \in \mathbb{R}^n$ are, respectively, an eigenvalue (undamped natural frequency squared) and the corresponding eigenvector. The modal matrix is commonly orthonormalized according to

$$U^{\mathrm{T}} M U = I_n. \tag{8}$$

Since Eq. (7) holds, it follows that

$$U^{\mathrm{T}} C U =: R, \quad U^{\mathrm{T}} K U = \mathrm{diag}[\omega_1^2, \omega_2^2, \ldots, \omega_n^2] =: \Omega^2, \tag{9}$$

where the symmetric matrix $R = [r_{jk}] = [u_j^{\mathrm{T}} C u_k] \in \mathbb{R}^{n \times n}$ is known as the modal damping matrix. Since C is a positive semi-definite matrix, the diagonal elements of R are non-negative.

In system (1), let

$$x(t) = Uq(t) \tag{10}$$

for all $t \geqslant 0$, where U is the modal matrix and

$$q(t) = [q_1(t)\ q_2(t)\ \ldots\ q_n(t)]^{\mathrm{T}} \in \mathbb{R}^n \tag{11}$$

is known as the modal co-ordinates. Using Eqs. (8) and (9), it is concluded that system (1) can be written as

$$\ddot{q}(t) + R\dot{q}(t) + \Omega^2 q(t) = \theta_n, \quad q(0) =: q_0 = U^{\mathrm{T}} M x_0, \quad \dot{q}(0) =: \dot{q}_0 = U^{\mathrm{T}} M \dot{x}_0 \tag{12}$$

for all $t \geqslant 0$, where R and Ω^2 are those in Eq. (9). System (12) is the representation of system (1) in the modal co-ordinates.

An easy-to-check condition for the existence of residual motion in system (1) is now given.

Theorem 2.5. *In system* (1), *let C be a positive semi-definite matrix. If*

$$CM^{-1}K = KM^{-1}C, \tag{13}$$

then system (1) *has residual motion.*

Proof. If Eq. (13) holds, then R in Eq. (12) is a diagonal matrix (see, e.g., Ref. [8, p. 144]), where the diagonal elements are non-negative. Since C is a positive semi-definite matrix, at least one diagonal element of R, say r_{ll}, is zero. Therefore, $q_l(\cdot)$ in Eq. (12) satisfies the following second order equation:

$$\ddot{q}_l(t) + \omega_l^2 q_l(t) = 0, \quad q_l(0) = q_{0_l}, \quad \dot{q}_l(0) = \dot{q}_{0_l} \tag{14}$$

for all $t \geqslant 0$. The solution of system (14), $t \mapsto q_l(t)$, is a non-decaying periodic function of time. This solution guarantees that $x(\cdot)$ in Eq. (10) would not tend to θ_n as $t \to 0$. That is, system (1) has residual solution. □

It is remarked that the condition in Eq. (13) is restrictive. There are examples where Eq. (13) does not hold, however, system (1) has residual motion. An example of such a system will be given later.

Next, a condition is given that guarantees the existence of residual motion for a large class of systems. In the following, the null space of a matrix L is denoted by $N(L)$.

Theorem 2.6. *In system* (1), *let C be a positive semi-definite matrix. If*

$$u_j \in N(C) \tag{15}$$

for at least one $j = 1, 2, \ldots, n$, where u_j *is the eigenvector in Eq.* (7), *then system* (1) *has residual motion.*

Proof. Let for a $j = l$, Eq. (15) hold. Thus, all elements on the lth row and the lth column of R in Eq. (12) are zero. Therefore, $q_l(\cdot)$ in Eq. (12) satisfies Eq. (14). The rest of the proof is similar to that of the proof of Theorem 2.5. □

In order to apply Theorem 2.6, first the eigenvector u_j in Eq. (7) is computed for all $j = 1, 2, \ldots, n$. Then, it is checked whether $Cu_j = \theta_n$ (equivalently, $u_j^{\mathrm{T}} Cu_j = 0$) for a $j = 1, 2, \ldots, n$.

A result to be used subsequently is as follows.

Lemma 2.7. *Let $C \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix. The equality $v^{\mathrm{T}} Cv = 0$ for a $v \in \mathbb{R}^n$ holds if and only if $v \in N(C)$. (Equivalently, the inequality $v^{\mathrm{T}} Cv > 0$ holds if and only if $v \notin N(C)$).*

Proof. See Appendix A. □

Next, conditions are given which guarantee the non-existence of residual motion in system (1) with small damping.

Theorem 2.8. *In system (1), let C be a positive semi-definite matrix and let $C = \varepsilon\tilde{C}$, where the real number $0 < \varepsilon \ll 1$. Moreover, let the natural frequencies of the system, ω_j, where $j = 1, 2, \ldots, n$, be distinct. System (1) does not have residual motion if and only if*

$$u_j \notin N(\tilde{C}) \tag{16}$$

for all $j = 1, 2, \ldots, n$.

Proof. ($\Leftarrow$) If system (1) does not have residual motion, then by Theorem 2.6, u_j cannot be in the null space of C (equivalently, $\tilde{C}$) for any $j = 1, 2, \ldots, n$.

($\Rightarrow$) In system (12), let $x(t) = p\exp(\lambda t)$ for all $t \geqslant 0$, where the vector $p \in \mathbb{C}^n$ and scalar $\lambda \in \mathbb{C}$. Then, the resulting eigenvalue problem is

$$(\lambda^2 M + \lambda\varepsilon\tilde{C} + K)p = \theta_n. \tag{17}$$

The solution of Eq. (17) is the eigenvalue $\lambda_j \in \mathbb{C}$ and the eigenvector $p_j \in \mathbb{C}^n$, where $j = 1, 2, \ldots, n$. The eigenvalue λ_j, for sufficiently small ε, satisfies (see Ref. [11])

$$\lambda_j = -\tfrac{1}{2}\varepsilon u_j^{\mathrm{T}}\tilde{C}u_j + \mathrm{i}(\omega_j + O(\varepsilon^2)) \tag{18}$$

for all $j = 1, 2, \ldots, n$, where ω_j is the undamped natural frequency satisfying Eq. (7) and $\mathrm{i} = \sqrt{-1}$. The conjugate of λ_j is obtained by changing $+\mathrm{i}$ to $-\mathrm{i}$ in Eq. (18). Since Eq. (16) holds, by Lemma 2.7, $u_j^{\mathrm{T}}\tilde{C}u_j > 0$, and the eigenvalue λ_j and its conjugate have negative real parts for all $j = 1, 2, \ldots, n$. Thus, $x(t) \to \theta_n$ as $t \to \infty$. □

In order to apply Theorem 2.8, first the eigenvector u_j in Eq. (7) is computed for all $j = 1, 2, \ldots, n$. Then, it is checked whether $\tilde{C}u_j \neq \theta_n$ (equivalently, $u_j^{\mathrm{T}}\tilde{C}u_j > 0$) for all $j = 1, 2, \ldots, n$.

3. Examples

In this section, several examples are given to illustrate the application of results obtained in this note in deciding the existence or non-existence of residual motion in system (1).

Example 3.1. This example is chosen from Ref. [7]. In system (1), let

$$M = I_4, \qquad C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \qquad K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}. \tag{19}$$

The damping matrix C is positive semi-definite, whereas the mass and stiffness matrices are positive definite. It can be easily verified that Eq. (13) holds. Thus, by Theorem 2.5, system (1) has residual motion. This fact can be corroborated by computing the eigenvalues of the matrix A in Eq. (4); the eigenvalues are

$$\pm 0.6180\mathrm{i}, \quad \pm 1.6180\mathrm{i}, \quad -1 \pm 0.6180\mathrm{i}, \quad -1 \pm 1.6180\mathrm{i}. \tag{20}$$

Two pairs of eigenvalues on the imaginary axis of the complex plane imply that the system has residual motion.

Example 3.2. Consider the system in Fig. 1. In this system, let

$$m_1 = m_2 = m_3 = 1, \quad c = 2, \quad k_1 = 1, \quad k_2 = 2, \quad k_3 = 3. \tag{21}$$

The coefficient matrices of this system are

$$M = I_3, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}. \tag{22}$$

The damping matrix C is positive semi-definite, whereas the mass and stiffness matrices are positive definite. It can be easily verified that Eq. (13) does not hold. Thus, Theorem 2.5 is not applicable.

The eigenvectors satisfying Eq. (7) for the system under consideration are

$$u_1 = \begin{bmatrix} -0.4415 \\ -0.6053 \\ -0.6623 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.8321 \\ 0 \\ -0.5547 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0.3358 \\ -0.7960 \\ 0.5036 \end{bmatrix}. \tag{23}$$

Having Cu_j computed for $j = 1, 2, 3$, it is concluded that $u_2 \in N(C)$. Thus, by Theorem 2.6, the system in Fig. 1 has residual motion. The eigenvalues of the matrix A in Eq. (4) for this system are

$$\pm 1.7321\mathrm{i}, \quad -0.4203 \pm 0.3473\mathrm{i}, \quad -0.5797 \pm 2.5283\mathrm{i}. \tag{24}$$

That is, the system has residual motion.

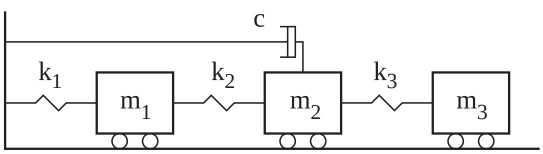

Fig. 1. A group of masses and springs. For parameter values in Eq. (21), the system has residual motion. If, however, the damping element is connected to the first mass m_1 or the last mass m_3, then there will be no residual motion.

Example 3.3. Consider the system in Example 3.1, except that the damping matrix is

$$C = 0.05 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{25}$$

The eigenvectors satisfying Eq. (7) for the system under consideration are

$$u_1 = \begin{bmatrix} 0.3717 \\ 0.6015 \\ 0.6015 \\ 0.3717 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -0.6015 \\ -0.3717 \\ 0.3717 \\ 0.6015 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -0.6015 \\ 0.3717 \\ 0.3717 \\ -0.6015 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -0.3717 \\ 0.6015 \\ -0.6015 \\ 0.3717 \end{bmatrix}. \tag{26}$$

It can be easily verified that $u_j^{\mathrm{T}} \tilde{C} u_j > 0$ for all $j = 1, 2, 3, 4$. Thus, by Theorem 2.8, system (1) does not have residual motion. The eigenvalues of the matrix A in Eq. (4) for this system are

$$-0.0026 \pm 0.6181\mathrm{i}, \quad -0.0026 \pm 1.1757\mathrm{i}, \quad -0.0474 \pm 1.6172\mathrm{i}, \quad -0.0474 + 1.9013\mathrm{i} \tag{27}$$

which corroborate the non-existence of residual motion.

4. Conclusions

In this note, linear n-degree-of-freedom dynamical systems, in which the mass and stiffness matrices are symmetric and positive definite and the damping matrix C is symmetric and positive semi-definite, are studied. Due to positive semi-definiteness of C, oscillations of the system components may not decay to zero, in which case the system is said to have residual motion. It can, however, happen that all components of the system come to rest even when C is positive semi-definite. The non-existence of residual motion is thus equivalent to the asymptotic stability of the system. This problem can be solved directly by computing all $2n$ eigenvalues of the matrix A in Eq. (4). In this note, conditions are given by which the existence or non-existence of residual motion is determined without computing the eigenvalues of A. In these conditions a major role is played by the eigenvectors corresponding to the (undamped) eigenvalue problem in Eq. (7): (1) if at least one eigenvector satisfying Eq. (7) belongs to the null space of C, then the system has residual motion; (2) a system with distinct natural frequencies does not have residual motion if and only if all eigenvectors satisfying Eq. (7) do not belong to the null space of C, and the elements of C are sufficiently small. It is conjectured that the latter statement is true even when the elements of C are not small.

The conditions presented in this note can be used to determine a small (or a minimum) number of damping elements which would constitute a positive semi-definite damping matrix, however, would render the system asymptotically stable (no residual motion). For instance, in Example 3.2 (see Fig. 1), if the damping element is connected to the first mass m_1 or the last mass m_3, then there will be no residual motion in the system.

Appendix A

Proof of Lemma 2.7. ($\Rightarrow$) If $v \in N(C)$, then $Cv = \theta_n$ and $v^{\mathrm{T}} C v = 0$.

($\Leftarrow$) Since C is a symmetric matrix, it has a complete set of orthogonal eigenvectors; see, e.g., Ref. [12, Theorems 3.1.2 and 3.1.3, p. 107]. Let $w_j \in \mathbb{R}^n$, where $j = 1, 2, \ldots, n$, denote an eigenvector of C. A vector $v \in \mathbb{R}^n$ can be written as

$$v = \sum_{j=1}^{n} \alpha_j w_j, \tag{A.1}$$

where $\alpha_j \in \mathbb{R}$ for all $j = 1, 2, \ldots, n$. Thus,

$$v^{\mathrm{T}} C v = \sum_{j=1}^{n} \lambda_j(C) \alpha_j^2, \tag{A.2}$$

where $\lambda_j(C)$ denotes the eigenvalue of the matrix C for all $j = 1, 2, \ldots, n$. Since C is a positive semi-definite matrix, all its eigenvalues are non-negative. If $v^{\mathrm{T}} C v = 0$, then from Eq. (A.2) it follows that $\alpha_j = 0$ for all j for which $\lambda_j(C) > 0$. Therefore,

$$Cv = C \sum_{j=1}^{n} \alpha_j w_j = \sum_{j=1}^{n} \alpha_j \lambda_j(C) w_j = \theta_n. \tag{A.3}$$

That is, $v \in N(C)$. $\square$

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